1901001101010001

EXAMINATION NOVEMBER 2024

MASTER OF ARTS (MATHEMATICS) (PART - I) (EXTERNAL)

MEASURE THEORY - LEVEL 1

[**Time: As Per Schedule]** **[Max. Marks: 100]**

Instructions:

1. Fill up strictly the following details on your answer book

a. Name of the Examination : **MASTER OF ARTS (MATHEMATICS) (PART –I) EXTERNAL**
b. Name of the Subject : **MEASURE THEORY-LEVEL 1**
c. Subject Code No : **1901001101010001**

2. Sketch neat and labelled diagram wherever necessary.
3. Figures to the right indicate full marks of the question.
4. Each question carries equal marks.
5. Figures to the right indicate full marks of the corresponding question.
6. Follow usual notations and conventions.

Seat No:

Student's Signature

Q.1

(a) Define outer measure and prove that outer measure of an interval is its length. **7**

(b) State and prove Fatau's lemma. **7**

(c) (i) Prove that a constant function defined over a measurable set is Measurable. **6**

(ii) If E and F are measurable sets such that F⊂E then prove that

$$m(E - F) = mE - mF.$$

OR

(a) If < E_n > is an infinite decreasing sequence of measurable sets and mE_1 is finite then prove that $\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} mE_n$. **7**

(b) Prove that (a, ∞) is measurable. **7**

(c) For a Riemann integrable bounded function f defined on $[a, b]$, prove that it is measurable and $R\int_a^b f(x)dx = \int_a^b f(x)dx$. **6**

Q.2 (a) Prove that the family of measurable sets is σ-algebra. **7**

(b) If $(f_n)$is a sequence of non-negative measurable function and $f_n(x) \to f(x)$ almost everywhere on a set E, then prove that $\int_E f \leq \underline{lim} \int_E f_n$ **7**

(c) Define upper and lower Riemann integral of a function. If $f(x) = \begin{cases} 0; x \text{ is rational} \\ 1: x \text{ is irrational} \end{cases}$ then show that $R\int_a^{\bar{b}} f(x)\, dx = b - a$ and $\int_{\underline{a}}^{b} f(x)dx = 0$. **6**

OR

(a) Let f be a nonnegative function which is integrable over a set E. Then show that for given ε > 0 there is δ > 0 such that for every set $A \subset E$ with $mA < \delta$ we have $\int_A f < \varepsilon$. **7**

(b) State and prove Lebesgue convergent theorem. **7**

(c) Let f be nonnegative function then show that $\int_E f = 0 \Longrightarrow f = 0$ almost everywhere. **6**

Q.3 (a) State and prove monotone convergence theorem. **7**

(b) Prove that a function $f \in BV([a, b])$if and only if f is a difference of two monotone real valued functions on a $[a, b]$. **7**

(c) If f is integrable on $[a, b]$and if $f \int_a^x f(t)\, dt = 0, \forall x \in [a, b]$,then prove that $f(t) = 0$ almost everywhere on $[a, b]$. **6**

OR

(a) If a function f is absolutely continuous in an interval and if $f'^{(x)} = 0$ almost everywhere, then prove that f is constant. **7**

(b) If f is bounded and measurable on $[a, b]$and $F(x) = \int_a^x f(t)dt + F(a)$then prove that $F'^{(x)} = f(x)$ for almost all $x \in [a, b]$. **7**

(c) If f is absolutely continuous on$[a, b]$. then prove that it is a function of bounded variation. **6**

Q.4 (a) If Y is a Banach space, then prove that $B(X, Y)$ is also a Banach space. **7**

(b) Prove that the space $\mathbb{C}^n$is complete. **7**

(c) Prove that compact subset of a metric space is closed and bounded. **6**

OR

(a) Show that the dual space of $l^1 is\ l^\infty$. **7**

(b) Define sequence space S and prove that S is a metric space. **7**

(c) Let $T: X \to Y$ and $S: Y \to Z$ be bijective linear operators, where X, Y, Z are vector spaces. Then prove that $(ST)^{-1}: Z \to$ X of$(ST)^{-1} = T^{-1}S^{-1}$. **6**

Q.5 (a) Prove that every finite dimensional normed space is complete. **7**

(b) Let $T: D(T) \to Y$ be a linear operator with $D(T) \subset X$ and$R(T) \subset Y$, where X and Y be real or complex vector space. Then prove that **7**

i) The inverse $T^{-1}: R(T) \longrightarrow D(T)$exists if and only if Tx = 0⇒x=0.

ii) If $T^{-1}: R(T) \longrightarrow D(T)$exists, it is linear.

(c) Let X, Y be vector spaces, both real or complex. Let $T: D(T) \to Y$ be a linear operator with domain $D(T) \subset X$ and range$R(T) \subset Y$. Then prove that the inverse $T^{-1}: R(T) \longrightarrow D(T)$exists if and only if $Tx = 0 \Rightarrow x = 0$. **6**

OR

(a) Show that the dual space of $l^p is\ l^q$; here, $1 < p < +\infty$ and q is the conjugate of p, that is $1/p$+$1/q = 1$. **7**

(b) Let $T: X \longrightarrow Y$ be a linear operator, where X and Y are vector space then prove that **7**

(i) The range$R(T)$is a vector space.

(ii) If dim $D(T) = n < \infty$, then dim $R(T) \leq n$.

(c) In an inner product space, if $x \perp y$ then prove that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. **6**

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